

# SQUARE FUNCTION ESTIMATES FOR DISCRETE RADON TRANSFORMS

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ABSTRACT. We show  $\ell^p(\mathbb{Z}^d)$  boundedness, for  $p \in (1, \infty)$ , of discrete singular integrals of Radon type with the aid of appropriate square function estimates, which can be thought as a discrete counterpart of the Littlewood–Paley theory. It is a very powerful approach which allows us to proceed as in the continuous case.

## 1. INTRODUCTION

Assume that  $K \in C^1(\mathbb{R}^k \setminus \{0\})$  is a Calderón–Zygmund kernel satisfying the differential inequality

$$(1.1) \quad |y|^k |K(y)| + |y|^{k+1} |\nabla K(y)| \leq 1$$

for all  $y \in \mathbb{R}^k$  with  $|y| \geq 1$  and the cancellation condition

$$(1.2) \quad \sup_{\lambda \geq 1} \left| \int_{1 \leq |y| \leq \lambda} K(y) dy \right| \leq 1.$$

Let  $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_{d_0}) : \mathbb{Z}^k \rightarrow \mathbb{Z}^{d_0}$  be a mapping where each component  $\mathcal{P}_j : \mathbb{Z}^k \rightarrow \mathbb{Z}$  is an integer-valued polynomial of  $k$  variables with  $\mathcal{P}_j(0) = 0$ . In the present article, as in [5], we are interested in the discrete singular Radon transform  $T^{\mathcal{P}}$  defined by

$$(1.3) \quad T^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y)$$

for a finitely supported function  $f : \mathbb{Z}^{d_0} \rightarrow \mathbb{R}$ . We prove the following.

**Theorem A.** *For every  $p \in (1, \infty)$  there is  $C_p > 0$  such that for all  $f \in \ell^p(\mathbb{Z}^{d_0})$*

$$(1.4) \quad \|T^{\mathcal{P}} f\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

*Moreover, the constant  $C_p$  is independent of the coefficients of the polynomial mapping  $\mathcal{P}$ .*

Theorem A was proven by Ionescu and Wainger in [5]. The operator  $T^{\mathcal{P}}$  is a discrete analogue of the continuous Radon transform  $R^{\mathcal{P}}$  defined by

$$(1.5) \quad R^{\mathcal{P}} f(x) = \text{p.v.} \int_{\mathbb{R}^k} f(x - \mathcal{P}(y)) K(y) dy.$$

Nowadays the operators  $R^{\mathcal{P}}$  and their  $L^p(\mathbb{R}^{d_0})$  boundedness properties for  $p \in (1, \infty)$  are very well understood. We refer to [8] for a detailed exposition and the references given there, and see also [4] for more general cases and more references. The key ingredient in proving  $L^p(\mathbb{R}^{d_0})$  bounds for  $R^{\mathcal{P}}$  is the Littlewood–Paley theory. More precisely, we begin with  $L^2(\mathbb{R}^{d_0})$  theory which, based on some oscillatory integral estimates for dyadic pieces of the multiplier corresponding to  $R^{\mathcal{P}}$ , provides bounds with acceptable decays. Then appealing to the Littlewood–Paley theory and interpolation it is possible to obtain general  $L^p(\mathbb{R}^{d_0})$  bounds for all  $p \in (1, \infty)$ . Now, one would like to follow the same scheme in the discrete case. However, the situation for  $T^{\mathcal{P}}$  is much more complicated due to arithmetic nature of this operator. Although  $\ell^2(\mathbb{Z}^{d_0})$  theory is based on estimates for some oscillatory integrals or rather exponential sums associated with dyadic pieces of the multiplier corresponding to  $T^{\mathcal{P}}$  as it was shown in [5], the  $\ell^p(\mathbb{Z}^{d_0})$  theory did not fall under the Littlewood–Paley paradigm as it is in the continuous case.

The main aim of this paper is to give a new proof of Theorem A using square function techniques. We construct a suitable square function which allows us to proceed as in the continuous case to obtain  $\ell^p(\mathbb{Z}^{d_0})$  theory for the operator (1.3). Our square function gives a new insight for these sort of problems (see especially [6] and [7]) and can be thought as a discrete counterpart of the Littlewood–Paley theory.

**1.1. Outline of the strategy of our proof.** Recall from [8, Chapter 6, §4.5, Chapter 13, §5.3] that given a kernel  $K$  satisfying (1.1) and (1.2) there are functions  $(K_n : n \geq 0)$  and a constant  $C > 0$  such that for  $x \neq 0$

$$(1.6) \quad K(x) = \sum_{n=0}^{\infty} K_n(x),$$

where for each  $n \geq 0$ , the kernel  $K_n$  is supported inside  $2^{n-2} \leq |x| \leq 2^n$ , satisfies

$$(1.7) \quad |x|^k |K_n(x)| + |x|^{k+1} |\nabla K_n(x)| \leq C$$

for all  $x \in \mathbb{R}^k$  such that  $|x| \geq 1$ , and has integral 0, provided that  $n \in \mathbb{N}$ . Thus in view of (1.7), instead of (1.4), it suffices to show that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that

$$(1.8) \quad \left\| \sum_{n \geq 0} T_n^{\mathcal{P}} f \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}$$

for all  $f \in \ell^p(\mathbb{Z}^d)$ , where

$$(1.9) \quad T_n^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{Q}(y)) K_n(y)$$

for each  $n \in \mathbb{Z}$ .

As we mentioned before the proof of inequality (1.8) will strongly follow the scheme of the proof of the corresponding inequality from the continuous setup. Now we describe the key points of our approach. To avoid some technicalities assume that  $\mathcal{P}(x) = (x^d, \dots, x)$  is a moment curve for some  $d = d_0 \geq 2$  and  $k = 1$ . Let  $m_n$  be the Fourier multiplier associated with the operator  $T_n^{\mathcal{P}}$ , i.e.  $T_n^{\mathcal{P}} f = \mathcal{F}^{-1}(m_n \hat{f})$ . As in [6] and [7] we introduce a family of appropriate projections  $(\Xi_n(\xi) : n \geq 0)$  which will localize asymptotic behaviour of  $m_n(\xi)$ . Namely, let  $\eta$  be a smooth bump function with a small support, fix  $l \in \mathbb{N}$  and define for each integer  $n \geq 0$  the following projections

$$(1.10) \quad \Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(\mathcal{E}_n^{-1}(\xi - a/q))$$

where  $\mathcal{E}_n$  is a diagonal  $d \times d$  matrix with positive entries  $(\varepsilon_j : 1 \leq j \leq d)$  such that  $\varepsilon_j \leq e^{-n^{1/5}}$  and

$$\mathcal{U}_{n^l} = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a = (a_1, \dots, a_d) \in \mathbb{N}_q^d \text{ and } \gcd(a_1, \dots, a_d, q) = 1 \text{ and } q \in P_{n^l}\}$$

for some family  $P_{n^l}$  such that  $\mathbb{N}_{n^l} \subseteq P_{n^l} \subseteq \mathbb{N}_{e^{n^{1/10}}}$ . All details are described in Section 2. Exploiting the ideas of Ionescu and Wainger from [5], we prove that for every  $p \in (1, \infty)$  there is a constant  $C_{l,p} > 0$  such that

$$(1.11) \quad \|\mathcal{F}^{-1}(\Xi_n \hat{f})\|_{\ell^p} \leq C_{l,p} \log(n+2) \|f\|_{\ell^p}.$$

Inequality (1.11) will be essential in our proof. Observe that (1.10) allows us to dominate (1.8) as follows

$$(1.12) \quad \left\| \sum_{n \geq 0} T_n^{\mathcal{P}} f \right\|_{\ell^p} \leq \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \right\|_{\ell^p} + \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n (1 - \Xi_n) \hat{f}) \right\|_{\ell^p}$$

and we can employ the ideas from the circle method of Hardy and Littlewood, which are implicit in the behaviour of the projections  $\Xi_n$  and  $1 - \Xi_n$ . Namely, the second norm on the right-hand side of (1.12) is bounded, since the multiplier  $m_n(1 - \Xi_n)$  is highly oscillatory. Thus appealing to (1.11) and a variant of Weyl's inequality with logarithmic decay which has been proven in [6], see Theorem 4, we can conclude that there is a constant  $C_p > 0$  such that for each  $n \geq 0$  we have

$$\|\mathcal{F}^{-1}(m_n (1 - \Xi_n) \hat{f})\|_{\ell^p} \leq C_p (n+1)^{-2} \|f\|_{\ell^p}.$$

Now the whole difficulty lies in proving

$$(1.13) \quad \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

For this purpose we construct new multipliers of the form

$$(1.14) \quad \Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}} (\eta(\mathcal{E}_{n+j}(\xi - a/q)) - \eta(\mathcal{E}_{n+j+1}(\xi - a/q))) \eta(\mathcal{E}_s(\xi - a/q))$$

such that

$$\Xi_n(\xi) \simeq \sum_{j \in \mathbb{Z}} \sum_{s \geq 0} \Delta_{n,s}^j(\xi).$$

Moreover, we will be able to show in Theorem 5, using Theorem 1, that for each  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that

$$(1.15) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \log(s+2) \|f\|_{\ell^p}.$$

for any  $s \geq 0$ , uniformly in  $j \in \mathbb{Z}$ . Estimate (1.15) can be thought as a discrete counterpart of the Littlewood–Paley inequality, see Theorem 5. This is the key ingredient in our proof which combined with the robust  $\ell^2(\mathbb{Z}^d)$  estimate

$$(1.16) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(m_n \Delta_{n,s}^j \hat{f})|^2 \right)^{1/2} \right\|_{\ell^2} \leq C 2^{-\varepsilon|j|} (s+1)^{-\delta_l} \|f\|_{\ell^2}$$

allows us to deduce (1.13). The last bound follows, since for each  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$  we have

$$m_n(\xi) = G(a/q) \Phi_n(\xi - a/q) + \mathcal{O}(2^{-n/2})$$

where  $G(a/q)$  is the Gaussian sum and  $\Phi_n$  is an integral counterpart of  $m_n$ , precise definitions can be found at beginning of Section 3. This observation leads to (1.16), because  $|G(a/q)| \leq Cq^{-\delta}$  and  $q \geq s^l$  if  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_{s^l}$ . The decay in  $|j|$  in (1.16) follows from the assumption on the support of  $\Xi_{n,s}^j$  and the behaviour of  $\Phi_n(\xi - a/q)$ , see Section 3 for more details.

The ideas of exploiting projections (1.10) have been initiated in [6] in the context of  $\ell^p(\mathbb{Z}^{d_0})$  boundedness of maximal functions corresponding respectively to the averaging Radon operators

$$(1.17) \quad M_N^{\mathcal{P}} f(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} f(x - \mathcal{P}(y))$$

where  $\mathbb{N}_N^k = \{1, 2, \dots, N\}^k$ , and the truncated singular Radon transforms

$$(1.18) \quad T_N^{\mathcal{P}} f(x) = \sum_{y \in \mathbb{B}_N \setminus \{0\}} f(x - \mathcal{P}(y)) K(y)$$

where  $\mathbb{B}_N = \{x \in \mathbb{Z}^k : |x| \leq N\}$ . These ideas, on the one hand, resulted in a new proof for Bourgain's maximal operators [1, 2, 3]. On the other hand, turned out to be flexible enough and attack  $\ell^p(\mathbb{Z}^{d_0})$  boundedness of maximal functions for operators with signs like in (1.18). In fact, in [6] we provided some vector-valued estimates for the maximal functions associated with (1.17) and (1.18). These estimates found applications in variational estimates for (1.17) and (1.18), which have been the subject of [7]. Our approach falls within the scope of a general scheme which has been recently developed in [6] and [7] and resulted in some unification in the theory of discrete analogues in harmonic analysis. The novelty of this paper is that it provides some counterpart of the Littlewood–Paley square function which is useful in the problems with arithmetic flavor. Furthermore, this square function theory is also an invaluable ingredient in the estimates of variational seminorm in [7].

The paper is organized as follows. In Section 2 we prove Theorem 1 which is essential in our approach, and guarantees (1.11). Ionescu and Wainger in [5] proved this result with  $(\log N)^D$  loss in norm where  $D > 0$  is a large power. In [6] we provided a slightly different proof and showed that  $\log N$  is possible. Moreover  $\log N$  loss is sharp for the method which we used. Since Theorem 1 is a deep theorem, which uses the most sophisticated tools developed to date in the field of discrete analogues, we have decided, for the sake of completeness, to provide necessary details. In Section 3 we prove Theorem A. To understand more quickly the proof of Theorem A, the reader may begin by looking at Section 3 first. These sections can be read independently, assuming the results from Section 2.

**1.2. Basic reductions.** We set

$$N_0 = \max\{\deg \mathcal{P}_j : 1 \leq j \leq d_0\}$$

and define the set

$$\Gamma = \{\gamma \in \mathbb{Z}^k \setminus \{0\} : 0 \leq \gamma_j \leq N_0 \text{ for each } j = 1, \dots, k\}$$

with the lexicographic order. Let  $d$  be the cardinality of  $\Gamma$ . Then we can identify  $\mathbb{R}^d$  with the space of all vectors whose coordinates are labeled by multi-indices  $\gamma \in \Gamma$ . Next we introduce the canonical polynomial mapping

$$\mathcal{Q} = (\mathcal{Q}_\gamma : \gamma \in \Gamma) : \mathbb{Z}^k \rightarrow \mathbb{Z}^d$$

where  $\mathcal{Q}_\gamma(x) = x^\gamma$  and  $x^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$ . The canonical polynomial mapping  $\mathcal{Q}$  determines anisotropic dilations. Namely, let  $A$  be a diagonal  $d \times d$  matrix such that

$$(Av)_\gamma = |\gamma|v_\gamma$$

for any  $v \in \mathbb{R}^d$  and  $\gamma \in \Gamma$ , where  $|\gamma| = \gamma_1 + \dots + \gamma_k$ . Then for every  $t > 0$  we set

$$t^A = \exp(A \log t)$$

i.e.  $t^A x = (t^{|\gamma|} x_\gamma : \gamma \in \Gamma)$  for  $x \in \mathbb{R}^d$  and we see that  $\mathcal{Q}(tx) = t^A \mathcal{Q}(x)$ .

Observe also that each  $\mathcal{P}_j$  can be expressed as

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma$$

for some  $c_j^\gamma \in \mathbb{R}$ . Moreover, the coefficients  $(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \dots, d_0\})$  define a linear transformation  $L : \mathbb{R}^d \rightarrow \mathbb{R}^{d_0}$  such that  $L\mathcal{Q} = \mathcal{P}$ . Indeed, it is enough to set

$$(Lv)_j = \sum_{\gamma \in \Gamma} c_j^\gamma v_\gamma$$

for each  $j \in \{1, \dots, d_0\}$  and  $v \in \mathbb{R}^d$ . Now instead of proving Theorem A we show the following.

**Theorem B.** *For every  $p \in (1, \infty)$  there is  $C_p > 0$  such that for all  $f \in \ell^p(\mathbb{Z}^d)$*

$$(1.19) \quad \|T^\mathcal{Q} f\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

In view of [8, Section 11] we can perform some lifting procedure which allows us to replace the underlying polynomial mapping  $\mathcal{P}$  from (1.4) by the canonical polynomial mapping  $\mathcal{Q}$ . Moreover, it shows that (1.19) implies (1.4) with the same constant  $C_p$ , see also [5] for more details. Therefore, the matters are reduced to proving (1.19) for the canonical polynomial mapping. The advantage of working with the canonical polynomial mapping  $\mathcal{Q}$  is that it has all coefficients equal to 1, and the uniform bound in this case is immediate. From now on for simplicity of notation we will write  $T = T^\mathcal{Q}$ .

**1.3. Notation.** Throughout the whole article  $C > 0$  will stand for a positive constant (possibly large constant) whose value may change from occurrence to occurrence. If there is an absolute constant  $C > 0$  such that  $A \leq CB$  ( $A \geq CB$ ) then we will write  $A \lesssim B$  ( $A \gtrsim B$ ). Moreover, we will write  $A \simeq B$  if  $A \lesssim B$  and  $A \gtrsim B$  hold simultaneously, and we will denote  $A \lesssim_\delta B$  ( $A \gtrsim_\delta B$ ) to indicate that the constant  $C > 0$  depends on some  $\delta > 0$ . Let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$  and for  $N \in \mathbb{N}$  we set

$$\mathbb{N}_N = \{1, 2, \dots, N\}.$$

For a vector  $x \in \mathbb{R}^d$  we will use the following norms

$$|x|_\infty = \max\{|x_j| : 1 \leq j \leq d\}, \quad \text{and} \quad |x| = \left( \sum_{j=1}^d |x_j|^2 \right)^{1/2}.$$

If  $\gamma$  is a multi-index from  $\mathbb{N}_0^k$  then  $|\gamma| = \gamma_1 + \dots + \gamma_k$ . Although, we use  $|\cdot|$  for the length of a multi-index  $\gamma \in \mathbb{N}_0^k$  and the Euclidean norm of  $x \in \mathbb{R}^d$ , their meaning will be always clear from the context and it will cause no confusions in the sequel.

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## 2. IONESCU-WAINGER TYPE MULTIPLIERS

For a function  $f \in L^1(\mathbb{R}^d)$  let  $\mathcal{F}$  denote the Fourier transform on  $\mathbb{R}^d$  defined as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x} dx.$$

If  $f \in \ell^1(\mathbb{Z}^d)$  we set

$$\hat{f}(\xi) = \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x}.$$

To simplify the notation we denote by  $\mathcal{F}^{-1}$  the inverse Fourier transform on  $\mathbb{R}^d$  and the inverse Fourier transform on the torus  $\mathbb{T}^d \equiv [0, 1)^d$  (Fourier coefficients). The meaning of  $\mathcal{F}^{-1}$  will be always clear from the context. Let  $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth function such that  $0 \leq \eta(x) \leq 1$  and

$$\eta(x) = \begin{cases} 1 & \text{for } |x| \leq 1/(16d), \\ 0 & \text{for } |x| \geq 1/(8d). \end{cases}$$

**Remark 2.1.** We will additionally assume that  $\eta$  is a convolution of two non-negative smooth functions  $\phi$  and  $\psi$  with compact supports contained in  $(-1/(8d), 1/(8d))^d$ .

This section is intended to prove Theorem 1, which is inspired by the ideas of Ionescu and Wainger from [5]. Let  $\rho > 0$  and for every  $N \in \mathbb{N}$  define

$$N_0 = \lfloor N^{\rho/2} \rfloor + 1 \quad \text{and} \quad Q_0 = (N_0!)^D$$

where  $D = D_\rho = \lfloor 2/\rho \rfloor + 1$ . Let  $\mathbb{P}_N = \mathbb{P} \cap (N_0, N]$  where  $\mathbb{P}$  is the set of all prime numbers. For any  $V \subseteq \mathbb{P}_N$  we define

$$\Pi(V) = \bigcup_{k \in \mathbb{N}_D} \Pi_k(V)$$

where for any  $k \in \mathbb{N}_D$

$$\Pi_k(V) = \{p_1^{\gamma_1} \cdots p_k^{\gamma_k} : \gamma_l \in \mathbb{N}_D \text{ and } p_l \in V \text{ are distinct for all } 1 \leq l \leq k\}.$$

In other words  $\Pi(V)$  is the set of all products of primes factors from  $V$  of length at most  $D$ , at powers between 1 and  $D$ . Now we introduce the sets

$$P_N = \{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Pi(\mathbb{P}_N) \cup \{1\}\}.$$

It is not difficult to see that every integer  $q \in \mathbb{N}_N$  can be uniquely written as  $q = Q \cdot w$  where  $Q|Q_0$  and  $w \in \Pi(\mathbb{P}_N) \cup \{1\}$ . Moreover, for sufficiently large  $N \in \mathbb{N}$  we have

$$q = Q \cdot w \leq Q_0 \cdot w \leq (N_0!)^D N^{D^2} \leq e^{N^\rho}$$

thus we have  $\mathbb{N}_N \subseteq P_N \subseteq \mathbb{N}_{e^{N^\rho}}$ . Furthermore, if  $N_1 \leq N_2$  then  $P_{N_1} \subseteq P_{N_2}$ .

For a subset  $S \subseteq \mathbb{N}$  we define

$$\mathcal{R}(S) = \{a/q \in \mathbb{T}^d \cap \mathbb{Q}^d : a \in A_q \text{ and } q \in S\}$$

where for each  $q \in \mathbb{N}$

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma))\}.$$

Finally, for each  $N \in \mathbb{N}$  we will consider the sets

$$(2.1) \quad \mathcal{U}_N = \mathcal{R}(P_N).$$

It is easy to see, if  $N_1 \leq N_2$  then  $\mathcal{U}_{N_1} \subseteq \mathcal{U}_{N_2}$ .

We will assume that  $\Theta$  is a multiplier on  $\mathbb{R}^d$  and for every  $p \in (1, \infty)$  there is a constant  $\mathbf{A}_p > 0$  such that for every  $f \in L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$  we have

$$(2.2) \quad \|\mathcal{F}^{-1}(\Theta \mathcal{F}f)\|_{L^p} \leq \mathbf{A}_p \|f\|_{L^p}.$$

For each  $N \in \mathbb{N}$  we define new periodic multipliers

$$(2.3) \quad \Delta_N(\xi) = \sum_{a/q \in \mathcal{U}_N} \Theta(\xi - a/q) \eta_N(\xi - a/q)$$

where  $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$  and  $\mathcal{E}_N$  is a diagonal  $d \times d$  matrix with positive entries  $(\varepsilon_\gamma : \gamma \in \Gamma)$  such that  $\varepsilon_\gamma \leq e^{-N^{2\rho}}$ . The main result is the following.

**Theorem 1.** *Let  $\Theta$  be a multiplier on  $\mathbb{R}^d$  obeying (2.2). Then for every  $\rho > 0$  and  $p \in (1, \infty)$  there is a constant  $C_{\rho,p} > 0$  such that for any  $N \in \mathbb{N}$  and  $f \in \ell^p(\mathbb{Z}^d)$*

$$(2.4) \quad \|\mathcal{F}^{-1}(\Delta_N \hat{f})\|_{\ell^p} \leq C_{\rho,p}(\mathbf{A}_p + 1) \log N \|f\|_{\ell^p}.$$

The main constructing blocks have been gathered in the next three subsections. Theorem 1 is a consequence of Theorem 3 and Proposition 2.2 proved below. To prove Theorem 1 we find some  $C_\rho > 0$  and disjoint sets  $\mathcal{U}_N^i \subseteq \mathcal{U}_N$  such that

$$\mathcal{U}_N = \bigcup_{1 \leq i \leq C_\rho \log N} \mathcal{U}_N^i$$

and we show that  $\Delta_N$  with the summation restricted to  $\mathcal{U}_N^i$  is bounded on  $\ell^p(\mathbb{Z}^d)$  for every  $p \in (1, \infty)$ . In order to construct  $\mathcal{U}_N^i$  we need a suitable partition of integers from the set  $\Pi(\mathbb{P}_N) \cup \{1\}$ , see also [5].

**2.1. Fundamental combinatorial lemma.** We begin with

**Definition 2.1.** A subset  $\Lambda \subseteq \Pi(V)$  has  $\mathcal{O}$  property if there is  $k \in \mathbb{N}_D$  and there are sets  $S_1, S_2, \dots, S_k$  with the following properties:

- (i) for each  $1 \leq j \leq k$  there is  $\beta_j \in \mathbb{N}$  such that  $S_j = \{q_{j,1}, \dots, q_{j,\beta_j}\}$ ;
- (ii) for every  $q_{j,s} \in S_j$  there are  $p_{j,s} \in V$  and  $\gamma_j \in \mathbb{N}_D$  such that  $q_{j,s} = p_{j,s}^{\gamma_j}$ ;
- (iii) for every  $w \in W$  there are unique numbers  $q_{1,s_1} \in S_1, \dots, q_{k,s_k} \in S_k$  such that  $w = q_{1,s_1} \dots q_{k,s_k}$ ;
- (iv) if  $(j, s) \neq (j', s')$  then  $(q_{j,s}, q_{j',s'}) = 1$ .

Now three comments are in order.

- (i) The set  $\Lambda = \{1\}$  has  $\mathcal{O}$  property corresponding to  $k = 0$ .
- (ii) If  $\Lambda$  has  $\mathcal{O}$  property, then each subset  $\Lambda' \subseteq \Lambda$  has  $\mathcal{O}$  property as well.
- (iii) If a set  $\Lambda$  has  $\mathcal{O}$  property then each element of  $\Lambda$  has the same number of prime factors  $k \leq D$ .

The main result is the following.

**Lemma 2.1.** *For every  $\rho > 0$  there exists a constant  $C_\rho > 0$  such that for every  $N \in \mathbb{N}$  the set  $\mathcal{U}_N$  can be written as a disjoint union of at most  $C_\rho \log N$  sets  $\mathcal{U}_N^i = \mathcal{R}(P_N^i)$  where*

$$(2.5) \quad P_N^i = \{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Lambda_i(\mathbb{P}_N)\}$$

and  $\Lambda_i(\mathbb{P}_N) \subseteq \Pi(\mathbb{P}_N) \cup \{1\}$  has  $\mathcal{O}$  property for each integer  $1 \leq i \leq C_\rho \log N$ .

*Proof.* We have to prove that for every  $V \subseteq \mathbb{P}_N$  the set  $\Pi(V)$  can be written as a disjoint union of at most  $C_k \log N$  sets with  $\mathcal{O}$  property. Fix  $k \in \mathbb{N}_D$ , let  $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{N}_D^k$  be a multi-index and observe that

$$\Pi_k(V) = \bigcup_{\gamma \in \mathbb{N}_D^k} \Pi_k^\gamma(V)$$

where

$$\Pi_k^\gamma(V) = \{p_1^{\gamma_1} \dots p_k^{\gamma_k} : p_l \in V \text{ are distinct for all } 1 \leq l \leq k\}.$$

Since there are  $D^k$  possible choices of exponents  $\gamma_1, \dots, \gamma_k \in \mathbb{N}_D$  when  $k \in \mathbb{N}_D$ , it only suffices to prove that every  $\Pi_k^\gamma(V)$  can be partitioned into a union (not necessarily disjoint) of at most  $C_k \log N$  sets with  $\mathcal{O}$  property.

We claim that for each  $k \in \mathbb{N}$  there is a constant  $C_k > 0$  and a family

$$(2.6) \quad \pi = \{\pi_i(V) : 1 \leq i \leq C_k \log |V|\}$$

of partitions of  $V$  such that

- (i) for every  $1 \leq i \leq C_k \log |V|$  each  $\pi_i(V) = \{V_1^i, \dots, V_k^i\}$  consists of pairwise disjoint subsets of  $V$  and  $V = V_1^i \cup \dots \cup V_k^i$ ;
- (ii) for every  $E \subseteq V$  with at least  $k$  elements there exists  $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$  such that  $E \cap V_j^i \neq \emptyset$  for every  $1 \leq j \leq k$ .

Assume for a moment we have constructed a family  $\pi$  as in (2.6). Then one sees that for a fixed  $\gamma \in \mathbb{N}_D^k$  we have

$$(2.7) \quad \Pi_k^\gamma(V) = \bigcup_{1 \leq i \leq C_k \log |V|} \Pi_{k,i}^\gamma(V)$$

where

$$\Pi_{k,i}^\gamma(V) = \{p_1^{\gamma_1} \dots p_k^{\gamma_k} : p_j^{\gamma_j} \in (V \cap V_j^i)^{\gamma_j} \text{ and } V_j^i \in \pi_i(V) \text{ for each } 1 \leq j \leq k\}$$

and  $(V \cap V_j^i)^{\gamma_j} = \{p^{\gamma_j} : p \in V \cap V_j^i\}$ . Indeed, the sum on the right-hand side of (2.7) is contained in  $\Pi_k^\gamma(V)$  since each  $\Pi_{k,i}^\gamma(V)$  is. For the opposite inclusion take  $p_1^{\gamma_1} \dots p_k^{\gamma_k} \in \Pi_k^\gamma(V)$  and let  $E = \{p_1, \dots, p_k\}$ , then property (ii) for the family (2.6) ensures that there is  $\pi_i(V) = \{V_1^i, \dots, V_k^i\} \in \pi$  such that  $E \cap V_j^i \neq \emptyset$  for every  $1 \leq j \leq k$ . Therefore,  $p_1^{\gamma_1} \dots p_k^{\gamma_k} \in \Pi_{k,i}^\gamma(V)$ . Furthermore, we see that for each  $1 \leq i \leq C_k \log N$  the sets  $\Pi_{k,i}^\gamma(V)$  have  $\mathcal{O}$  property.

The proof will be completed if we construct the family  $\pi$  as in (2.6) for the set  $V$ . We assume, for simplicity, that  $V = \mathbb{N}_N$  but the result is true for all  $V \subseteq \mathbb{N}_N$  containing at least  $k$  elements. Now it will be more comfortable to work with surjective mappings  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$  rather than with partitions of  $\mathbb{N}_N$  into  $k$  non-empty subsets. It will cause no changes to us, since every surjection  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$  determines a partition  $\{f^{-1}[\{m\}] : 1 \leq m \leq k\}$  of  $\mathbb{N}_N$  into  $k$  non-empty subsets.

For the proof we employ a probabilistic argument. Indeed, let  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$  be a random surjective mapping. Assume that for every  $n \in \mathbb{N}_N$  and  $m \in \mathbb{N}_k$  we have  $\mathbb{P}(\{f(n) = m\}) = 1/k$  independently of all other  $n \in \mathbb{N}_N$ . For every  $E \subseteq \mathbb{N}_N$  with  $k$  elements we have  $\mathbb{P}(\{|f[E]| = k\}) = k!/k^k$ . It suffices to show that for some  $r \simeq_k \log N$  and  $f_1, \dots, f_r$  random surjections we have

$$\mathbb{P}(\{\forall E \subseteq \mathbb{N}_N \ |E| = k \ \exists_{1 \leq l \leq r} \ |f_l[E]| = k\}) > 0.$$

In other words, for each  $E \subseteq \mathbb{N}_N$  with cardinality  $k$  it is always possible to find, with a positive probability, among at most  $C_k \log N$  random surjections at least one  $f : \mathbb{N}_N \mapsto \mathbb{N}_k$  such that  $|f[E]| = k$ . Then the set  $\{f^{-1}[\{m\}] : 1 \leq m \leq k\}$  is a partition of  $\mathbb{N}_N$  and  $E \cap f^{-1}[\{m\}] \neq \emptyset$  for every  $1 \leq m \leq k$ .

The task now is to determine the exact value of  $r \simeq_k \log N$ . Take now  $1 \leq r \leq N$  independent random surjections  $f_1, \dots, f_r$  and observe that

$$\begin{aligned} \mathbb{P}(\{\exists E \subseteq \mathbb{N}_N \ |E| = k \ \forall_{1 \leq l \leq r} \ |f_l[E]| < k\}) &\leq \sum_{E \subseteq \mathbb{N}_N : |E|=k} \mathbb{P}(\{\forall_{1 \leq l \leq r} \ |f_l[E]| < k\}) \\ &= \sum_{E \subseteq \mathbb{N}_N : |E|=k} \left(1 - \frac{k!}{k^k}\right)^r = \binom{N}{k} \left(1 - \frac{k!}{k^k}\right)^r \leq \left(\frac{eN}{k}\right)^k e^{-r \frac{k!}{k^k}} = e^{k \log(\frac{eN}{k}) - r \frac{k!}{k^k}}. \end{aligned}$$

Therefore

$$\mathbb{P}(\{\exists E \subseteq \mathbb{N}_N \ |E| = k \ \forall_{1 \leq l \leq r} \ |f_l[E]| < k\}) < 1$$

if and only if

$$r > \frac{k^{k+1}}{k!} \log \left( \frac{eN}{k} \right).$$

Thus taking

$$r = \left\lceil \frac{k^{k+1}}{k!} \log \left( \frac{eN}{k} \right) \right\rceil + 1 \simeq C_k \log N$$

we see that it does the job. This completes the proof of Lemma 2.1.  $\square$

**2.2. Further reductions and square function estimates.** Now we can write

$$\Delta_N = \sum_{1 \leq i \leq C_\rho \log N} \Delta_N^i$$

where for each  $1 \leq i \leq C_\rho \log N$

$$(2.8) \quad \Delta_N^i(\xi) = \sum_{a/q \in \mathcal{U}_N^i} \Theta(\xi - a/q) \eta_N(\xi - a/q)$$

with  $\mathcal{U}_N^i$  as in Lemma 2.1. The proof of Theorem 1 will be completed if we show that for every  $p \in (1, \infty)$  and  $\rho > 0$ , there is a constant  $C > 0$  such that for any  $N \in \mathbb{N}$  and  $1 \leq i \leq C_\rho \log N$  we have

$$(2.9) \quad \|\mathcal{F}^{-1}(\Delta_N^i \hat{f})\|_{\ell^p} \leq C(\mathbf{A}_p + 1) \|f\|_{\ell^p}$$

for every  $f \in \ell^p(\mathbb{Z}^d)$ .

Let

$$(2.10) \quad \Lambda \subseteq \Pi(\mathbb{P}_N) \cup \{1\}$$

be a set with  $\mathcal{O}$  property, see Definition 2.1. Define

$$\mathcal{U}_N^\Lambda = \mathcal{R}(\{q = Q \cdot w : Q|Q_0 \text{ and } w \in \Lambda\})$$



and  $\mathscr{W}_N = \mathcal{R}(\Lambda)$ , and we introduce

$$(2.11) \quad \Delta_N^\Lambda(\xi) = \sum_{a/q \in \mathscr{W}_N^\Lambda} \Theta(\xi - a/q) \eta_N(\xi - a/q).$$

We show that for every  $p \in (1, \infty)$  and  $\rho > 0$ , there is a constant  $C > 0$  such that for any  $N \geq 8^{\max\{p, p'\}/\rho}$  and for any set  $\Lambda$  as in (2.10) and for every  $f \in \ell^p(\mathbb{Z}^d)$  we have

$$(2.12) \quad \|\mathcal{F}^{-1}(\Delta_N^\Lambda \hat{f})\|_{\ell^p} \leq C(\mathbf{A}_p + 1) \|f\|_{\ell^p}.$$

For  $N \leq 8^{\max\{p, p'\}/\rho}$  the bound in (2.12) is obvious, since we allow the constant  $C > 0$  to depend on  $p$  and  $\rho$ . Moreover, by the duality and interpolation, it suffices to prove (2.12) for  $p = 2r$  where  $r \in \mathbb{N}$ . If  $\Lambda = \Lambda_i(P_N)$  as in Lemma 2.1 for some  $1 \leq i \leq C_\rho \log N$ , then we see that  $\mathscr{W}_N^\Lambda = \mathscr{W}_N^i$  and  $\Delta_N^\Lambda = \Delta_N^i$ , and consequently (2.12) implies (2.9) as desired.

The function  $\Theta(\xi) \eta_N(\xi)$  is regarded as a periodic function on  $\mathbb{T}^d$ , thus

$$\begin{aligned} \Delta_N^\Lambda(\xi) &= \sum_{a/q \in \mathscr{W}_N^\Lambda} \Theta(\xi - a/q) \eta_N(\xi - a/q) \\ &= \sum_{b \in \mathbb{N}_{Q_0}^d} \sum_{a/w \in \mathscr{W}_N} \Theta(\xi - b/Q_0 - a/w) \eta_N(\xi - b/Q_0 - a/w) \end{aligned}$$

where we have used the fact, that if  $(q_1, q_2) = 1$  then for every  $a \in \mathbb{Z}^d$ , there are unique  $a_1, a_2 \in \mathbb{Z}^d$ , such that  $a_1/q_1, a_2/q_2 \in [0, 1)^d$  and

$$(2.13) \quad \frac{a}{q_1 q_2} = \frac{a_1}{q_1} + \frac{a_2}{q_2} \pmod{\mathbb{Z}^d}.$$

Since  $\Lambda$  has  $\mathcal{O}$  property then according to Definition 2.1 there is an integer  $1 \leq k \leq 2/\rho + 1$  and there are sets  $S_1, \dots, S_k$  such that for any  $j \in \mathbb{N}_k$  we have  $S_j = \{q_{j,1}, \dots, q_{j,\beta_j}\}$  for some  $\beta_j \in \mathbb{N}$ .

Now for each  $j \in \mathbb{N}_k$  we introduce

$$\mathcal{U}_{\{j\}} = \{a_{j,s}/q_{j,s} \in \mathbb{T}^d \cap \mathbb{Q}^d : s \in \mathbb{N}_{\beta_j} \text{ and } a_{j,s} \in A_{q_{j,s}}\}$$

and for any  $M = \{j_1, \dots, j_m\} \subseteq \mathbb{N}_k$  let

$$\mathcal{U}_M = \{u_{j_1} + \dots + u_{j_m} \in \mathbb{T}^d \cap \mathbb{Q}^d : u_{j_l} \in \mathcal{U}_{\{j_l\}} \text{ for any } l \in \mathbb{N}_m\}.$$

For any sequence  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$  determined by the set  $M$ , let us define

$$\mathcal{V}_M^\sigma = \{a_{j_1, s_{j_1}}/q_{j_1, s_{j_1}} + \dots + a_{j_m, s_{j_m}}/q_{j_m, s_{j_m}} \in \mathbb{T}^d \cap \mathbb{Q}^d : a_{j_l, s_{j_l}} \in A_{q_{j_l, s_{j_l}}} \text{ for any } l \in \mathbb{N}_m\}.$$

Note that  $\mathcal{V}_M^\sigma$  is a subset of  $\mathcal{U}_M$  with fixed denominators  $q_{j_1, s_{j_1}}, \dots, q_{j_m, s_{j_m}}$ . If  $M = \emptyset$  then we have  $\mathcal{U}_M = \mathcal{V}_M = \{0\}$ . Let

$$\chi(\xi) = \mathbb{1}_\Lambda(\xi) \quad \text{and} \quad \Omega_N(\xi) = \Theta(\xi) \eta_N(\xi).$$

Then again by (2.13) we obtain

$$\begin{aligned} (2.14) \quad \Delta_N^\Lambda(\xi) &= \sum_{a/w \in \mathscr{W}_N} \sum_{b \in \mathbb{N}_{Q_0}^d} \Theta(\xi - b/Q_0 - a/w) \eta_N(\xi - b/Q_0 - a/w) \\ &= \sum_{s_1 \in \mathbb{N}_{\beta_1}} \sum_{a_{1,s_1} \in A_{q_{1,s_1}}} \dots \sum_{s_k \in \mathbb{N}_{\beta_k}} \sum_{a_{k,s_k} \in A_{q_{k,s_k}}} m_{a_{1,s_1}/q_{1,s_1} + \dots + a_{k,s_k}/q_{k,s_k}}(\xi) = \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} m_u(\xi) \end{aligned}$$

where

$$(2.15) \quad m_u(\xi) = m_{a_{1,s_1}/q_{1,s_1} + \dots + a_{k,s_k}/q_{k,s_k}}(\xi) = \chi(q_{1,s_1} \dots q_{k,s_k}) \sum_{b \in \mathbb{N}_{Q_0}^d} \Omega_N\left(\xi - b/Q_0 - \sum_{j=1}^k a_{j,s_j}/q_{j,s_j}\right)$$

for  $u = a_{1,s_1}/q_{1,s_1} + \dots + a_{k,s_k}/q_{k,s_k}$ .

From now on we will write, for every  $u \in \mathcal{U}_{\mathbb{N}_k}$ , that

$$(2.16) \quad f_u(x) = \mathcal{F}^{-1}(m_u \hat{f})(x)$$

with  $f \in \ell^{2r}(\mathbb{Z}^d)$  and  $r \in \mathbb{N}$ . Therefore,

$$(2.17) \quad \mathcal{F}^{-1}(\Delta_N^\Lambda \hat{f})(x) = \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u(x)$$



and the proof of inequality (2.12) will follow from the following.

**Theorem 2.** *Suppose that  $\rho > 0$  and  $r \in \mathbb{N}$  are given. Then there is a constant  $C_{\rho,r} > 0$  such that for any  $N > 8^{2r/\rho}$  and for any set  $\Lambda$  as in (2.10) and for every  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have*

$$(2.18) \quad \left\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \right\|_{\ell^{2r}} \leq C_{\rho,r} \|f\|_{\ell^{2r}}.$$

Moreover, the integer  $k \in \mathbb{N}_D$ , the set  $\mathcal{U}_{\mathbb{N}_k}$  and consequently the sets  $S_1, \dots, S_k$  are determined by the set  $\Lambda$  as it was described above.

The estimate (2.18) will follow from Theorem 3 and Proposition 2.2 formulated below. Let us introduce a suitable square function which will be useful in bounding (2.18). For any  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$  and any sequence  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  let us define the following square function  $\mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$  associated with the sequence  $(f_u : u \in \mathcal{U}_{\mathbb{N}_k})$  of complex-valued functions as in (2.16), by setting

$$(2.19) \quad \mathcal{S}_{L,M}^\sigma(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) = \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{u \in \mathcal{U}_{M \setminus L}} \sum_{v \in \mathcal{V}_L^\sigma} f_{w+u+v}(x) \right|^2 \right)^{1/2},$$

where  $M^c = \mathbb{N}_k \setminus M$ . For some  $s_{j_i} \in \{s_{j_1}, \dots, s_{j_l}\}$  we will write

$$\left\| \mathcal{S}_{L,M}^\sigma(f_u(x) : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell_{s_{j_i}}^2} = \left( \sum_{s_{j_i} \in \mathbb{N}_{\beta_{j_i}}} |\mathcal{S}_{L,M}^{(s_{j_1}, \dots, s_{j_l})}(f_u : u \in \mathcal{U}_{\mathbb{N}_k})(x)|^2 \right)^{1/2}$$

which defines some function which depends on  $x \in \mathbb{Z}^d$  and on each  $s_{j_n} \in \{s_{j_1}, \dots, s_{j_l}\} \setminus \{s_{j_i}\}$ .

For the proof of (2.18) we have to exploit the fact that the Fourier transform of  $f_u$  is defined as a sum of disjointly supported smooth cut-off functions. Then appropriate subsums of  $\sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u$  should be strongly orthogonal to each other.

Theorem 2 will be proved as a consequence of Theorem 3 and Proposition 2.2 below.

**Theorem 3.** *Suppose that  $\rho > 0$  and  $r \in \mathbb{N}$  are given. Then there is a constant  $C_{\rho,r} > 0$  such that for any  $N > 8^{2r/\rho}$  and for any set  $\Lambda$  as in (2.10) and for every  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have*

$$(2.20) \quad \left\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \right\|_{\ell^{2r}}^{2r} \leq C_{\rho,r} \sum_{\substack{M \subseteq \mathbb{N}_k \\ M = \{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \left\| \mathcal{S}_{M,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r}.$$

Moreover, the integer  $k \in \mathbb{N}_D$ , the set  $\mathcal{U}_{\mathbb{N}_k}$  and consequently the sets  $S_1, \dots, S_k$  are determined by the set  $\Lambda$  as it was described above the formulation of Theorem 2.

*Proof.* Under the assumptions of Theorem 2, there is a constant  $C_r > 0$  such that for any  $M \subseteq \mathbb{N}_k$  and  $L = \{j_1, \dots, j_l\} \subseteq M$  and  $j_n \in M \setminus L$  and for any  $\sigma = (s_{j_1}, \dots, s_{j_l}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}}$  determined by the set  $L$  we have

$$(2.21) \quad \left\| \mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}} \leq C_r \left\| \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell_{s_{j_n}}^2} \right\|_{\ell^{2r}}$$

where  $\sigma \oplus s_{j_n} = (s_{j_1}, \dots, s_{j_l}, s_{j_n}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_l}} \times \mathbb{N}_{\beta_{j_n}}$  is the sequence determined by the set  $L \cup \{s_{j_n}\}$ . Moreover, the right-hand side in (2.21) can be controlled in the following way

$$(2.22) \quad \left\| \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell_{s_{j_n}}^2} \right\|_{\ell^{2r}}^{2r} \leq C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\ + C_r \left\| \mathcal{S}_{L, M \setminus \{j_n\}}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r}.$$

The proof of (2.21) and (2.22) can be found in [6]. Therefore, (2.21) combined with (2.22) yields

$$(2.23) \quad \left\| \mathcal{S}_{L,M}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \leq C_r \sum_{s_{j_n} \in \mathbb{N}_{\beta_{j_n}}} \left\| \mathcal{S}_{L \cup \{j_n\}, M}^{\sigma \oplus s_{j_n}}(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\ + C_r \left\| \mathcal{S}_{L, M \setminus \{j_n\}}^\sigma(f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r}.$$

Applying (2.23) recursively we obtain

$$\begin{aligned}
(2.24) \quad & \left\| \sum_{u \in \mathcal{U}_{\mathbb{N}_k}} f_u \right\|_{\ell^{2r}}^{2r} = \left\| \mathcal{S}_{\emptyset, \mathbb{N}_k} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\
& \lesssim_r \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \left\| \mathcal{S}_{\{k\}, \mathbb{N}_k}^{(s_k)} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} + \left\| \mathcal{S}_{\emptyset, \mathbb{N}_{k-1}} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\
& \lesssim_r \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \left\| \mathcal{S}_{\{k-1, k\}, \mathbb{N}_k}^{(s_{k-1}, s_k)} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} + \sum_{s_k \in \mathbb{N}_{\beta_{j_k}}} \left\| \mathcal{S}_{\{k\}, \mathbb{N}_k \setminus \{k-1\}}^{(s_k)} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\
& \quad + \sum_{s_{k-1} \in \mathbb{N}_{\beta_{j_{k-1}}}} \left\| \mathcal{S}_{\{k-1\}, \mathbb{N}_{k-1}}^{(s_{k-1})} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} + \left\| \mathcal{S}_{\emptyset, \mathbb{N}_{k-2}} (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \\
& \lesssim_r \cdots \lesssim_{\rho, r} \sum_{M=\{j_1, \dots, j_m\}} \sum_{\substack{M \subseteq \mathbb{N}_k \\ \sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}}} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{v \in \mathcal{V}_M^\sigma} f_{w+v}(x) \right|^2 \right)^r.
\end{aligned}$$

The proof of (2.20) is completed.  $\square$

**2.3. Concluding remarks and the proof of Theorem 2.** Now Theorem 3 reduces the proof of inequality (2.18) to showing the following estimate

$$(2.25) \quad \sum_{\substack{M \subseteq \mathbb{N}_k \\ M=\{j_1, \dots, j_m\}}} \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \left\| \mathcal{S}_{M, M}^\sigma (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} \lesssim_r \|f\|_{\ell^{2r}}^{2r}$$

for any  $f \in \ell^{2r}(\mathbb{Z}^d)$  which is a characteristic function of a finite set in  $\mathbb{Z}^d$ . Firstly, we prove the following.

**Proposition 2.2.** *Under the assumptions of Theorem 2, there exists a constant  $C_{\rho, r} > 0$  such that for any  $M = \{j_1, \dots, j_m\} \subseteq \mathbb{N}_k$  any  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$  determined by the set  $M$  and  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have*

$$(2.26) \quad \left\| \mathcal{S}_{M, M}^\sigma (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}} \leq C_{\rho, r} \mathbf{A}_r \left\| \mathcal{S}_{M, M}^\sigma \left( \mathcal{F}^{-1} \left( \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - u) \hat{f}(\xi) \right) : u \in \mathcal{U}_{\mathbb{N}_k} \right) \right\|_{\ell^{2r}}.$$

*Proof.* We assume, without of loss of generality, that  $N \in \mathbb{N}$  is large. Let  $B_h = q_{j_1, s_{j_1}} \cdots q_{j_m, s_{j_m}} \cdot Q_0 \leq e^{N^\rho}$  and observe that according to the notation from (2.16) and (2.14), we have

$$\begin{aligned}
(2.27) \quad & \left\| \mathcal{S}_{M, M}^\sigma (f_u : u \in \mathcal{U}_{\mathbb{N}_k}) \right\|_{\ell^{2r}}^{2r} = \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \sum_{v \in \mathcal{V}_M^\sigma} f_{w+v}(x) \right|^2 \right)^r \\
& \leq \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1} \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \Theta(\xi - b/Q_0 - v - w) \eta_N(\xi - b/Q_0 - v - w) \hat{f}(\xi) \right) (x) \right|^2 \right)^r \\
& \quad = \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1}(\Theta \eta_N G(\cdot; n, w))(B_h x + n) \right|^2 \right)^r
\end{aligned}$$

where

$$(2.28) \quad G(\xi; n, w) = \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}^d} \hat{f}(\xi + b/Q_0 + v + w) e^{-2\pi i(b/Q_0 + v) \cdot n}.$$

We know that for each  $0 < p < \infty$  there is a constant  $C_p > 0$  such that for any  $d \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_d \in \mathbb{C}^d$  we have

$$(2.29) \quad \left( \int_{\mathbb{C}^d} |\lambda_1 z_1 + \dots + \lambda_d z_d|^p e^{-\pi |z|^2} dz \right)^{1/p} = C_p (|\lambda_1|^2 + \dots + |\lambda_d|^2)^{1/2}.$$

By Proposition 4.5 from [6], with the sequence of multipliers  $\Theta_N = \Theta$  for all  $N \in \mathbb{N}$  and  $\Theta$  as in (2.2), we have

$$(2.30) \quad \left\| \mathcal{F}^{-1}(\Theta \eta_N G(\cdot; n, w))(B_h x + n) \right\|_{\ell^{2r}(x)} \leq C_{\rho, r} \mathbf{A}_{2r} \left\| \mathcal{F}^{-1}(\eta_N G(\cdot; n, w))(B_h x + n) \right\|_{\ell^{2r}(x)}$$

since  $\inf_{\gamma \in \Gamma} \varepsilon_{\gamma}^{-1} \geq e^{N^{2\rho}} \geq 2e^{(d+1)N^\rho} \geq B_h$  for sufficiently large  $N \in \mathbb{N}$ .

Therefore, combining (2.30) with (2.29) we obtain that

$$\begin{aligned}
(2.31) \quad & \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |\mathcal{F}^{-1}(\Theta \eta_N G(\cdot; n, w))(B_h x + n)|^2 \right)^r \\
&= C_{2r}^{2r} \int_{\mathbb{C}^d} \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left| \mathcal{F}^{-1} \left( \Theta \eta_N \left( \sum_{w \in \mathcal{U}_{M^c}} z_w G(\cdot; n, w) \right) \right) (B_h x + n) \right|^{2r} e^{-\pi|z|^2} dz \\
&\lesssim_r \int_{\mathbb{C}^d} \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left| \mathcal{F}^{-1} \left( \sum_{w \in \mathcal{U}_{M^c}} z_w \eta_N G(\cdot; n, w) \right) (B_h x + n) \right|^{2r} e^{-\pi|z|^2} dz \\
&\lesssim_r \sum_{n \in \mathbb{N}_{B_h}^d} \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} |\mathcal{F}^{-1}(\eta_N G(\cdot; n, w))(B_h x + n)|^2 \right)^r \\
&\lesssim_r \sum_{x \in \mathbb{Z}^d} \left( \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1} \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \hat{f}(\xi) \right) (x) \right|^2 \right)^r.
\end{aligned}$$

This completes the proof of Proposition 2.2.  $\square$

Now we are able to finish the proof of Theorem 2.

*Proof of Theorem 2.* It remains to show that there exists a constant  $C_{\rho, r} > 0$  such that for any  $M = \{j_1, \dots, j_m\} \subseteq \mathbb{N}_k$  any  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$  determined by the set  $M$  and  $f \in \ell^{2r}(\mathbb{Z}^d)$  we have

$$(2.32) \quad \sum_{\sigma \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}} \left\| \mathcal{S}_{M, M}^\sigma \left( \mathcal{F}^{-1} \left( \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - u) \hat{f}(\xi) \right) : u \in \mathcal{U}_{\mathbb{N}_k} \right) \right\|_{\ell^{2r}}^{2r} \leq C_{\rho, r}^{2r} \|f\|_{\ell^{2r}}^{2r}.$$

Since there are  $2^k$  possible choices of sets  $M \subseteq \mathbb{N}_k$  and  $k \in \mathbb{N}_D$  then (2.25) will follow and the proof of Theorem 2 will be completed. If  $r = 1$  then Plancherel's theorem does the job since the functions  $\eta_N(\xi - b/Q_0 - v - w)$  are disjointly supported for all  $b/Q_0 \in \mathbb{N}_{Q_0}$ ,  $w \in \mathcal{U}_{M^c}$ ,  $v \in \mathcal{V}_M^\sigma$  and  $\sigma = (s_{j_1}, \dots, s_{j_m}) \in \mathbb{N}_{\beta_{j_1}} \times \dots \times \mathbb{N}_{\beta_{j_m}}$ . For general  $r \geq 2$ , since  $\|f\|_{\ell^{2r}}^{2r} = \|f\|_{\ell^2}^2$  because we have assumed that  $f$  is a characteristic function of a finite set in  $\mathbb{Z}^d$ , it suffices to prove for any  $x \in \mathbb{Z}^d$  that

$$(2.33) \quad \sum_{w \in \mathcal{U}_{M^c}} \left| \mathcal{F}^{-1} \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \hat{f}(\xi) \right) (x) \right|^2 \leq C_{\rho, r}.$$

In fact, since  $\|f\|_{\ell^\infty} = 1$ , it is enough to show

$$(2.34) \quad \left\| \mathcal{F}^{-1} \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \right) \right\|_{\ell^1} \leq C_{\rho, r}$$

for any sequence of complex numbers  $(\alpha(w) : w \in \mathcal{U}_{M^c})$  such that

$$(2.35) \quad \sum_{w \in \mathcal{U}_{M^c}} |\alpha(w)|^2 = 1.$$

Computing the Fourier transform we obtain

$$\begin{aligned}
(2.36) \quad & \mathcal{F}^{-1} \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} \eta_N(\xi - b/Q_0 - v - w) \right) \\
&= \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i x \cdot w} \right) \cdot \det(\mathcal{E}_N) \mathcal{F}^{-1} \eta(\mathcal{E}_N x) \cdot \left( \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} e^{-2\pi i x \cdot (b/Q_0 + v)} \right).
\end{aligned}$$

The function

$$(2.37) \quad \sum_{v \in \mathcal{V}_M^\sigma} \sum_{b \in \mathbb{N}_{Q_0}} e^{-2\pi i x \cdot (b/Q_0 + v)}$$

can be written as a sum of  $2^m$  functions

$$(2.38) \quad \sum_{b \in \mathbb{N}_Q} e^{-2\pi i x \cdot (b/Q)} = \begin{cases} Q^d & \text{if } x \equiv 0 \pmod{Q}, \\ 0 & \text{otherwise,} \end{cases}$$

where possible values of  $Q$  are products of  $Q_0$  and  $p_{j_i, s_{j_i}}^{\gamma_i}$  or  $p_{j_i, s_{j_i}}^{\gamma_i-1}$  for  $i \in \mathbb{N}_m$ . Therefore, the proof of (2.34) will be completed if we show that

$$(2.39) \quad \left\| \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i Q x \cdot w} \right) \cdot Q^d \det(\mathcal{E}_N) \mathcal{F}^{-1} \eta(Q \mathcal{E}_N x) \right\|_{\ell^1(x)} \leq C_{\rho, r}$$

for any integer  $Q \leq e^{N^\rho}$  such that  $(Q, q_{j,s}) = 1$ , for all  $j \in M^c$  and  $s \in \mathbb{N}_{\beta_j}$ .

Recall that, according Remark 2.1, in our case  $\eta = \phi * \psi$  for some two smooth functions  $\phi, \psi$  supported in  $(-1/(8d), 1/(8d))^d$ . Therefore, by the Cauchy-Schwarz inequality we only need to prove that

$$(2.40) \quad Q^{d/2} \det(\mathcal{E}_N)^{1/2} \left\| \mathcal{F}^{-1} \phi(Q \mathcal{E}_N x) \right\|_{\ell^2(x)} \leq C_{\rho, r}$$

and

$$(2.41) \quad Q^{d/2} \det(\mathcal{E}_N)^{1/2} \left\| \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i Q x \cdot w} \right) \cdot \mathcal{F}^{-1} \psi(Q \mathcal{E}_N x) \right\|_{\ell^2(x)} \leq C_{\rho, r}.$$

Since  $(Q, q_{j,s}) = 1$ , for all  $j \in M^c$  and  $s \in \mathbb{N}_{\beta_j}$  then  $Qw \notin \mathbb{Z}^d$  for any  $w \in \mathcal{U}_{M^c}$  and its denominator is bounded by  $N^D$ . We can assume, without loss of generality, that  $Qw \in [0, 1)^d$  by the periodicity of  $x \mapsto e^{-2\pi i x \cdot Qw}$ . Inequality (2.40) easily follows from Plancherel's theorem. In order to prove (2.41) observe that by the change of variables one has

$$\left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i x \cdot Qw} \right) \cdot \mathcal{F}^{-1} \psi(Q \mathcal{E}_N x) = Q^{-d} \det(\mathcal{E}_N)^{-1} \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) \mathcal{F}^{-1} (\psi(Q^{-1} \mathcal{E}_N^{-1}(\cdot - Qw)))(x).$$

Therefore, Plancherel's theorem and the last identity yield

$$(2.42) \quad Q^d \det(\mathcal{E}_N) \left\| \left( \sum_{w \in \mathcal{U}_{M^c}} \alpha(w) e^{-2\pi i Q x \cdot w} \right) \cdot \mathcal{F}^{-1} \psi(Q \mathcal{E}_N x) \right\|_{\ell^2(x)}^2 \\ = \sum_{w \in \mathcal{U}_{M^c}} |\alpha(w)|^2 \int_{\mathbb{R}^d} |\psi(\xi - \mathcal{E}_N^{-1} w)|^2 d\xi + \sum_{\substack{w_1, w_2 \in \mathcal{U}_{M^c} \\ w_1 \neq w_2}} \alpha(w_1) \overline{\alpha(w_2)} \int_{\mathbb{R}^d} \psi(\xi) \overline{\psi(\xi - \mathcal{E}_N^{-1}(w_1 - w_2))} d\xi.$$

The first sum on the right-hand side of (2.42) is bounded in view of (2.35). The second one vanishes since the function  $\psi$  is supported in  $(-1/(8d), 1/(8d))^d$  and  $|\mathcal{E}_N^{-1}(w_1 - w_2)|_\infty \geq e^{N^{2\rho}} N^{-2D} > 1$ , for sufficiently large  $N$ . The proof of Theorem 2 is completed.  $\square$

### 3. PROOF OF THEOREM B

This section is intended to provide the proof of Theorem B. In fact, in view of the decomposition of the kernel  $K$  into dyadic pieces as in (1.6), instead of inequality (1.19), it suffices to show that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for all  $f \in \ell^p(\mathbb{Z}^d)$  we have

$$(3.1) \quad \left\| \sum_{n \in \mathbb{Z}} T_n f \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}$$

where

$$(3.2) \quad T_n f(x) = \sum_{y \in \mathbb{Z}^k} f(x - \mathcal{Q}(y)) K_n(y)$$

with the kernel  $K_n$  as in (1.6) for each  $n \in \mathbb{Z}$ .

**3.1. Exponential sums and  $\ell^2(\mathbb{Z}^d)$  approximations.** Recall that for  $q \in \mathbb{N}$

$$A_q = \{a \in \mathbb{N}_q^d : \gcd(q, (a_\gamma : \gamma \in \Gamma))\}.$$

Now for  $q \in \mathbb{N}$  and  $a \in A_q$  we define the Gaussian sums

$$G(a/q) = q^{-k} \sum_{y \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(y)}.$$

Let us observe that there exists  $\delta > 0$  such that

$$(3.3) \quad |G(a/q)| \lesssim q^{-\delta}.$$

This follows from the multi-dimensional variant of Weyl's inequality (see [9, Proposition 3]).

Let  $P$  be a polynomial in  $\mathbb{R}^k$  of degree  $d \in \mathbb{N}$  such that

$$P(x) = \sum_{0 < |\gamma| \leq d} \xi_\gamma x^\gamma.$$

Given  $N \geq 1$ , let  $\Omega_N$  be a convex set in  $\mathbb{R}^k$  such that

$$\Omega_N \subseteq \{x \in \mathbb{R}^k : |x - x_0| \leq cN\}$$

for some  $x_0 \in \mathbb{R}^k$  and  $c > 0$ . We define the Weyl sums

$$(3.4) \quad S_N = \sum_{n \in \Omega_N \cap \mathbb{Z}^k} e^{2\pi i P(n)} \varphi(n)$$

where  $\varphi : \mathbb{R}^k \mapsto \mathbb{C}$  is a continuously differentiable function which for some  $C > 0$  satisfies

$$(3.5) \quad |\varphi(x)| \leq C, \quad \text{and} \quad |\nabla \varphi(x)| \leq C(1 + |x|)^{-1}.$$

In [6] we have proven Theorem 4 which is a refinement of the estimates for the multi-dimensional Weyl sums  $S_N$ , where the limitations  $N^\varepsilon \leq q \leq N^{k-\varepsilon}$  from [9, Proposition 3] are replaced by the weaker restrictions  $(\log N)^\beta \leq q \leq N^k (\log N)^{-\beta}$  for appropriate  $\beta$ . Namely.

**Theorem 4.** *Assume that there is a multi-index  $\gamma_0$  such that  $0 < |\gamma_0| \leq d$  and*

$$\left| \xi_{\gamma_0} - \frac{a}{q} \right| \leq \frac{1}{q^2}$$

*for some integers  $a, q$  such that  $0 \leq a \leq q$  and  $(a, q) = 1$ . Then for any  $\alpha > 0$  there is  $\beta_\alpha > 0$  so that, for any  $\beta \geq \beta_\alpha$ , if*

$$(3.6) \quad (\log N)^\beta \leq q \leq N^{|\gamma_0|} (\log N)^{-\beta}$$

*then there is a constant  $C > 0$*

$$(3.7) \quad |S_N| \leq CN^k (\log N)^{-\alpha}.$$

*The implied constant  $C$  is independent of  $N$ .*

Let  $(m_n : n \geq 0)$  be a sequence of multipliers on  $\mathbb{T}^d$ , corresponding to the operators (3.2). Then for any finitely supported function  $f : \mathbb{Z}^d \mapsto \mathbb{C}$  we see that

$$T_n f(x) = \mathcal{F}^{-1}(m_n \hat{f})(x)$$

where

$$m_n(\xi) = \sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y).$$

For  $n \geq 0$  we set

$$\Phi_n(\xi) = \int_{\mathbb{R}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) dy.$$

Using multi-dimensional version of van der Corput's lemma (see [10, Proposition 2.1]) we obtain

$$(3.8) \quad |\Phi_n(\xi)| \lesssim \min \{1, |2^{nA} \xi|_\infty^{-1/d}\}.$$

Moreover, if  $n \geq 1$  we have

$$(3.9) \quad |\Phi_n(\xi)| = \left| \Phi_n(\xi) - \int_{\mathbb{R}^k} K_n(y) dy \right| \lesssim \min \{1, |2^{nA} \xi|_\infty\}.$$

The next proposition shows relations between  $m_n$  and  $\Phi_n$ .

**Proposition 3.1.** *There is a constant  $C > 0$  such that for every  $n \in \mathbb{N}$  and for every  $\xi \in [1/2, 1/2)^d$  satisfying*

$$\left| \xi_\gamma - \frac{a_\gamma}{q} \right| \leq L_1^{-|\gamma|} L_2$$

*for all  $\gamma \in \Gamma$ , where  $1 \leq q \leq L_3 \leq 2^{n/2}$ ,  $a \in A_q$ ,  $L_1 \geq 2^n$  and  $L_2 \geq 1$  we have*

$$(3.10) \quad |m_n(\xi) - G(a/q) \Phi_n(\xi - a/q)| \leq C \left( L_3 2^{-n} + L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n / L_1)^{|\gamma|} \right) \leq C L_2 L_3 2^{-n}.$$

*Proof.* Let  $\theta = \xi - a/q$ . For any  $r \in \mathbb{N}_q^k$ , if  $y \equiv r \pmod{q}$  then for each  $\gamma \in \Gamma$

$$\xi_\gamma y^\gamma \equiv \theta_\gamma y^\gamma + (a_\gamma/q) r^\gamma \pmod{1},$$

thus

$$\xi \cdot \mathcal{Q}(y) \equiv \theta \cdot \mathcal{Q}(y) + (a/q) \cdot \mathcal{Q}(r) \pmod{1}.$$

Therefore,

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) = \sum_{r \in \mathbb{N}_q^k} e^{2\pi i (a/q) \cdot \mathcal{Q}(r)} \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy+r)} K_n(qy+r).$$

If  $2^{n-2} \leq |qy+r|, |qy| \leq 2^n$  then by the mean value theorem we obtain

$$|\theta \cdot \mathcal{Q}(qy+r) - \theta \cdot \mathcal{Q}(qy)| \lesssim |r| \sum_{\gamma \in \Gamma} |\theta_\gamma| \cdot 2^{n(|\gamma|-1)} \lesssim q \sum_{\gamma \in \Gamma} L_1^{-|\gamma|} L_2 2^{n(|\gamma|-1)} \lesssim L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}$$

and

$$|K_n(qy+r) - K_n(qy)| \lesssim 2^{-n(k+1)} L_3.$$

Thus

$$\sum_{y \in \mathbb{Z}^k} e^{2\pi i \xi \cdot \mathcal{Q}(y)} K_n(y) = G(a/q) \cdot q^k \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) + \mathcal{O}\left(L_3 2^{-n} + L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\right).$$

Now one can replace the sum on the right-hand side by the integral. Indeed, again by the mean value theorem we obtain

$$\begin{aligned} & \left| \sum_{y \in \mathbb{Z}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) - \int_{\mathbb{R}^k} e^{2\pi i \theta \cdot \mathcal{Q}(qt)} K_n(qt) dt \right| \\ &= \left| \sum_{y \in \mathbb{Z}^k} \int_{[0,1]^k} (e^{2\pi i \theta \cdot \mathcal{Q}(qy)} K_n(qy) - e^{2\pi i \theta \cdot \mathcal{Q}(q(y+t))} K_n(q(y+t))) dt \right| \\ &= \mathcal{O}\left(q^{-k} L_3 2^{-n} + q^{-k} L_2 L_3 2^{-n} \sum_{\gamma \in \Gamma} (2^n/L_1)^{|\gamma|}\right). \end{aligned}$$

This completes the proof of Proposition 3.1.  $\square$

**3.2. Discrete Littlewood–Paley theory.** Fix  $j, n \in \mathbb{Z}$  and  $N \in \mathbb{N}$  and let  $\mathcal{E}_N$  be a diagonal  $d \times d$  matrix with positive entries  $(\varepsilon_\gamma : \gamma \in \Gamma)$  such that  $\varepsilon_\gamma \leq e^{-N^{2\rho}}$  with  $\rho > 0$  as in Section 2. Let us consider the multipliers

$$(3.11) \quad \Omega_N^{j,n}(\xi) = \sum_{a/q \in \mathcal{U}_N} \Phi_{j,n}(\xi - a/q) \eta_N(\xi - a/q)$$

with  $\eta_N(\xi) = \eta(\mathcal{E}_N^{-1}\xi)$  and  $\Phi_{j,n}(\xi) = \Phi(2^{nA+jI}\xi)$ , where  $\Phi$  is a Schwartz function such that  $\Phi(0) = 0$ .

If we had  $\mathcal{U}_N = \{0\}$  then  $\Omega_N^{j,n}(\xi)$  could be treated as a standard Littlewood–Paley projector. Now we formulate an abstract theorem which can be thought as a discrete variant of Littlewood–Paley theory. Its proof will be based on Theorem 1. Here we only obtain some square function estimate which is interesting in its own right. However, we will be able appreciate its usefulness later, in the proof of inequality (3.1).

**Theorem 5.** *For every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for all  $-\infty \leq M_1 \leq M_2 \leq \infty$ ,  $j \in \mathbb{Z}$  and  $N \in \mathbb{N}$  and every  $f \in \ell^p(\mathbb{Z}^d)$  we have*

$$(3.12) \quad \left\| \left( \sum_{M_1 \leq n \leq M_2} |\mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \log N \|f\|_{\ell^p}.$$

*Proof.* By Khinchine's inequality (3.12) is equivalent to the following

$$(3.13) \quad \left( \int_0^1 \left\| \sum_{M_1 \leq n \leq M_2} \varepsilon_n(t) \mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f}) \right\|_{\ell^p}^p dt \right)^{1/p} \lesssim \log N \|f\|_{\ell^p}.$$

Observe that the multiplier from (3.13) can be rewritten as follows

$$\sum_{M_1 \leq n \leq M_2} \varepsilon_n(t) \Omega_N^{j,n}(\xi) = \sum_{a/q \in \mathcal{U}_N} \sum_{M_1 \leq n \leq M_2} \mathbf{m}_n(\xi - a/q) \eta_N(\xi - a/q)$$

with the functions

$$\mathbf{m}_n(\xi) = \varepsilon_n(t) \Phi(2^{nA+jI}\xi).$$

We observe that

$$|\mathbf{m}_n(\xi)| \lesssim \min \{ |2^{nA+jI}\xi|_\infty, |2^{nA+jI}\xi|_\infty^{-1} \}.$$

The first bound follows from the mean-value theorem, since

$$|\Phi(2^{nA+jI}\xi)| = |\Phi(2^{nA+jI}\xi) - \Phi(0)| \lesssim |2^{nA+jI}\xi| \sup_{\xi \in \mathbb{R}^d} |\nabla \Phi(\xi)| \lesssim |2^{nA+jI}\xi|_\infty.$$

The second bound follows since  $\Phi$  is a Schwartz function. Moreover, for every  $p \in (1, \infty)$  there is  $C_p > 0$  such that

$$\left\| \sup_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\mathbf{m}_n \mathcal{F} f)| \right\|_{L^p} \leq C_p \|f\|_{L^p}$$

for every  $f \in L^p(\mathbb{R}^d)$ . Therefore, by [8] the multiplier

$$\sum_{M_1 \leq n \leq M_2} \mathbf{m}_n(\xi)$$

corresponds to a continuous singular integral, thus defines a bounded operator on  $L^p(\mathbb{R}^d)$  for all  $p \in (1, \infty)$  with the bound independent of  $j \in \mathbb{Z}$  and  $-\infty \leq M_1 \leq M_2 \leq \infty$ . Hence, Theorem 1 applies and the multiplier

$$\sum_{M_1 \leq n \leq M_2} \varepsilon_n(t) \Omega_N^{j,n}(\xi)$$

defines a bounded operator on  $\ell^p(\mathbb{Z}^d)$  with the  $\log N$  loss, and (3.13) is established.  $\square$

**Remark 3.1.** If the function  $\Phi$  is a real-valued function then we have

$$(3.14) \quad \left\| \sum_{M_1 \leq n \leq M_2} \mathcal{F}^{-1}(\Omega_N^{j,n} \hat{f}_n) \right\|_{\ell^p} \leq C_p \log N \left\| \left( \sum_{M_1 \leq n \leq M_2} |f_n|^2 \right)^{1/2} \right\|_{\ell^p}.$$

This is the dual version of inequality (3.12) for any sequence of functions  $(f_n : M_1 \leq n \leq M_2)$  such that

$$\left\| \left( \sum_{M_1 \leq n \leq M_2} |f_n|^2 \right)^{1/2} \right\|_{\ell^p} < \infty.$$

We have gathered all necessary ingredients to prove inequality (3.1).

*Proof of inequality (3.1).* Let  $\chi > 0$  and  $l \in \mathbb{N}$  be the numbers whose precise valued will be adjusted later. As in [6] we will consider for every  $n \in \mathbb{N}_0$  the multipliers

$$(3.15) \quad \Xi_n(\xi) = \sum_{a/q \in \mathcal{U}_n} \eta(2^{n(A-\chi I)}(\xi - a/q))$$

with  $\mathcal{U}_N$  which has been defined in Section 2. Theorem 1 yields, for every  $p \in (1, \infty)$ , that

$$(3.16) \quad \left\| \mathcal{F}^{-1}(\Xi_n \hat{f}) \right\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}.$$

The implicit constant in (3.16) depends on  $\rho > 0$  from Theorem 1. From now on we will assume that  $l \in \mathbb{N}$  and  $\rho > 0$  are related by the equation

$$(3.17) \quad 10\rho l = 1.$$

Assume that  $f : \mathbb{Z}^d \mapsto \mathbb{C}$  has finite support and  $f \geq 0$ . Observe that

$$(3.18) \quad \left\| \sum_{n \geq 0} T_n f \right\|_{\ell^p} \leq \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n \hat{f}) \right\|_{\ell^p} + \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n(1 - \Xi_n) \hat{f}) \right\|_{\ell^p}.$$

Without of loss of generality we may assume that  $p \geq 2$ , the case  $1 < p \leq 2$  follows by the duality then.



**3.3. The estimate of the second norm in (3.18).** It suffices to show that

$$(3.19) \quad \|\mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f})\|_{\ell^p} \lesssim (n+1)^{-2}\|f\|_{\ell^p}.$$

For this purpose we define for every  $x \in \mathbb{Z}^d$  the Radon averages

$$M_N f(x) = N^{-k} \sum_{y \in \mathbb{N}_N^k} f(x - \mathcal{Q}(y)).$$

From [6] follows that for every  $p \in (1, \infty)$  there is a constant  $C_p > 0$  such that for every  $f \in \ell^p(\mathbb{Z}^d)$  we have

$$(3.20) \quad \left\| \sup_{N \in \mathbb{N}} |M_N f| \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

Then for every  $1 < p < \infty$  by (3.16) and (3.20) we obtain

$$(3.21) \quad \|\mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f})\|_{\ell^p} \leq \left\| \sup_{N \in \mathbb{N}} M_N f \right\|_{\ell^p} + \left\| \sup_{N \in \mathbb{N}} M_N (|\mathcal{F}^{-1}(\Xi_n \hat{f})|) \right\|_{\ell^p} \lesssim \log(n+2)\|f\|_{\ell^p}$$

since we have a pointwise bound

$$(3.22) \quad |\mathcal{F}^{-1}(m_n \hat{f})(x)| = |T_n f(x)| \lesssim M_{2^n} f(x).$$

We show that it is possible to improve estimate (3.21) for  $p = 2$ . Indeed, by Theorem 4 we will show that for big enough  $\alpha > 0$ , which will be specified later, and for all  $n \in \mathbb{N}_0$  we have

$$(3.23) \quad |m_n(\xi)(1 - \Xi_n(\xi))| \lesssim (n+1)^{-\alpha}.$$

By Dirichlet's principle we have for every  $\gamma \in \Gamma$

$$|\xi_\gamma - a_\gamma/q_\gamma| \leq q_\gamma^{-1} n^\beta 2^{-n|\gamma|}$$

where  $1 \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ . In order to apply Theorem 4 we must show that there exists some  $\gamma \in \Gamma$  such that  $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ . Suppose for a contradiction that for every  $\gamma \in \Gamma$  we have  $1 \leq q_\gamma < n^\beta$  then for some  $q \leq \text{lcm}(q_\gamma : \gamma \in \Gamma) \leq n^{\beta d}$  we have

$$|\xi_\gamma - a'_\gamma/q| \leq n^\beta 2^{-n|\gamma|}$$

where  $\gcd(q, \gcd(a'_\gamma : \gamma \in \Gamma)) = 1$ . Hence, taking  $a' = (a'_\gamma : \gamma \in \Gamma)$  we have  $a'/q \in \mathcal{U}_{n^l}$  provided that  $\beta d < l$ . On the other hand, if  $1 - \Xi_n(\xi) \neq 0$  then for every  $a'/q \in \mathcal{U}_{n^l}$  there exists  $\gamma \in \Gamma$  such that

$$|\xi_\gamma - a'_\gamma/q| > (16d)^{-1} 2^{-n(|\gamma| - \chi)}.$$

Therefore

$$2^{\chi n} < 16dn^\beta$$

but this is impossible when  $n \in \mathbb{N}$  is large. Hence, there is  $\gamma \in \Gamma$  such that  $n^\beta \leq q_\gamma \leq n^{-\beta} 2^{n|\gamma|}$ . Thus by Theorem 4

$$|m_n(\xi)| \lesssim (n+1)^{-\alpha}$$

provided that  $1 - \Xi_n(\xi) \neq 0$ . This yields (3.23) and we obtain

$$(3.24) \quad \|\mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f})\|_{\ell^2} \lesssim (1+n)^{-\alpha} \log(n+2)\|f\|_{\ell^2}.$$

Interpolating (3.24) with (3.21) we obtain

$$(3.25) \quad \|\mathcal{F}^{-1}(m_n(1 - \Xi_n)\hat{f})\|_{\ell^p} \lesssim (1+n)^{-c_p \alpha} \log(n+2)\|f\|_{\ell^p}.$$

for some  $c_p > 0$ . Choosing  $\alpha > 0$  and  $l \in \mathbb{N}$  appropriately large one obtains (3.19).

**3.4. The estimate of the first norm in (3.18).** Note that for any  $\xi \in \mathbb{T}^d$  so that

$$|\xi_\gamma - a_\gamma/q| \leq 2^{-n(|\gamma|-\chi)}$$

for every  $\gamma \in \Gamma$  with  $1 \leq q \leq e^{n^{1/10}}$  we have

$$(3.26) \quad m_n(\xi) = G(a/q)\Phi_n(\xi - a/q) + q^{-\delta}E_{2^n}(\xi)$$

where

$$(3.27) \quad |E_{2^n}(\xi)| \lesssim 2^{-n/2}.$$

Proposition 3.1, with  $L_1 = 2^n$ ,  $L_2 = 2^{\chi n}$  and  $L_3 = e^{n^{1/10}}$ , establishes (3.26) and (3.27), since for sufficiently large  $n \in \mathbb{N}$  we have

$$q^\delta |E_{2^n}(\xi)| \lesssim q^\delta L_2 L_3 2^{-n} \lesssim (e^{-n((1-\chi)\log 2 - 2n^{-9/10})}) \lesssim 2^{-n/2}$$

provided  $\chi > 0$  is sufficiently small. Now for every  $j, n \in \mathbb{N}_0$  we introduce the multipliers

$$(3.28) \quad \Xi_n^j(\xi) = \sum_{a/q \in \mathcal{U}_{n^l}} \eta(2^{nA+jI}(\xi - a/q))^2$$

and we note that

$$(3.29) \quad \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n^j \hat{f}) \right\|_{\ell^p} \leq \left\| \sum_{n \geq 0} \mathcal{F}^{-1} \left( \sum_{-\chi n \leq j < n} m_n (\Xi_n^j - \Xi_n^{j+1}) \hat{f} \right) \right\|_{\ell^p} \\ + \left\| \sum_{n \geq 0} \mathcal{F}^{-1}(m_n \Xi_n^j \hat{f}) \right\|_{\ell^p} = I_p^1 + I_p^2.$$

We will estimate  $I_p^1$  and  $I_p^2$  separately. For this purpose observe that by (3.26) and (3.27) for every  $a/q \in \mathcal{U}_{n^l}$  we have

$$(3.30) \quad |m_n(\xi)| \lesssim q^{-\delta} |\Phi_n(\xi - a/q)| + q^{-\delta} |E_{2^n}(\xi)| \\ \lesssim q^{-\delta} (\min \{1, |2^{nA}(\xi - a/q)|_\infty, |2^{nA}(\xi - a/q)|_\infty^{-1/d}\} + 2^{-n/2})$$

where the last inequality follows from (3.8) and (3.9). Therefore by (3.30) we get

$$(3.31) \quad |m_n(\xi)(\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2)| \lesssim q^{-\delta} (2^{-|j|/d} + 2^{-n/2}).$$

**3.4.1. Bounding  $I_p^2$ .** It will suffice to show, for some  $\varepsilon = \varepsilon_p > 0$ , that

$$(3.32) \quad \|\mathcal{F}^{-1}(m_n \Xi_n^n \hat{f})\|_{\ell^p} \lesssim 2^{-\varepsilon n} \|f\|_{\ell^p}.$$

Observe that for any  $1 < p < \infty$  by (3.22), (3.20) and (3.16) we have

$$(3.33) \quad \|\mathcal{F}^{-1}(m_n \Xi_n^n \hat{f})\|_{\ell^p} \leq \left\| \sup_{N \in \mathbb{N}} M_N(|\mathcal{F}^{-1}(\Xi_n^n \hat{f})|) \right\|_{\ell^p} \lesssim \|\mathcal{F}^{-1}(\Xi_n^n \hat{f})\|_{\ell^p} \lesssim \log(n+2) \|f\|_{\ell^p}.$$

For  $p = 2$  by Plancherel's theorem and by (3.30) we obtain

$$(3.34) \quad \|\mathcal{F}^{-1}(m_n \Xi_n^n \hat{f})\|_{\ell^2} = \left( \int_{\mathbb{T}^d} \sum_{a/q \in \mathcal{U}_{n^l}} |m_n(\xi)|^2 \eta(2^{nA+nI}(\xi - a/q))^4 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \lesssim 2^{-n/(2d)} \|f\|_{\ell^2}.$$

Therefore, interpolating (3.33) with (3.34) we obtain for every  $p \in (1, \infty)$  that

$$\|\mathcal{F}^{-1}(m_n \Xi_n^n \hat{f})\|_{\ell^p} \lesssim 2^{-\varepsilon n} \|f\|_{\ell^p}$$

which in turn implies (3.32) and  $I_p^2 \lesssim \|f\|_{\ell^p}$ .

3.4.2. *Bounding  $I_p^1$ .* Define for any  $0 \leq s < n$  new multipliers

$$\Delta_{n,s}^j(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} (\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2) \eta(2^{s(A-\chi I)}(\xi - a/q))^2$$

and we observe that by the definition (3.28) we have

$$\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi) = \sum_{0 \leq s < n} \Delta_{n,s}^j(\xi).$$

Moreover,

$$\begin{aligned} \eta(2^{nA+jI}(\xi))^2 - \eta(2^{nA+(j+1)I}(\xi))^2 \\ = (\eta(2^{nA+jI}(\xi))^2 - \eta(2^{nA+(j+1)I}(\xi))^2) \cdot (\eta(2^{nA+(j-1)I}(\xi)) - \eta(2^{nA+(j+2)I}(\xi))). \end{aligned}$$

Thus we see

$$\Delta_{n,s}^j(\xi) = \Delta_{n,s}^{j,1}(\xi) \cdot \Delta_{n,s}^{j,2}(\xi),$$

where

$$\Delta_{n,s}^{j,1}(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} (\eta(2^{nA+(j-1)I}(\xi - a/q)) - \eta(2^{nA+(j+2)I}(\xi - a/q))) \eta(2^{s(A-\chi I)}(\xi - a/q))$$

and

$$\Delta_{n,s}^{j,2}(\xi) = \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} (\eta(2^{nA+jI}(\xi - a/q))^2 - \eta(2^{nA+(j+1)I}(\xi - a/q))^2) \eta(2^{s(A-\chi I)}(\xi - a/q)).$$

Moreover,  $\Delta_{n,s}^{j,1}$  and  $\Delta_{n,s}^{j,2}$  are the multipliers which satisfy the assumptions of Theorem 5. Therefore,

$$\begin{aligned} (3.35) \quad I_p^1 &= \left\| \sum_{n \geq 0} \mathcal{F}^{-1} \left( \sum_{-\chi n \leq j < n} \sum_{0 \leq s < n} \Delta_{n,s}^{j,1} m_n \Delta_{n,s}^{j,2} \hat{f} \right) \right\|_{\ell^p} \\ &\leq \sum_{s \geq 0} \sum_{j \in \mathbb{Z}} \left\| \sum_{n \geq \max\{j, -j/\chi, s\}} \mathcal{F}^{-1} \left( \Delta_{n,s}^{j,1} m_n \Delta_{n,s}^{j,2} \hat{f} \right) \right\|_{\ell^p} \\ &\lesssim \sum_{s \geq 0} \sum_{j \in \mathbb{Z}} \log s \left\| \left( \sum_{n \geq \max\{j, -j/\chi, s\}} |\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p}. \end{aligned}$$

In the last step we have used (3.14). The task now is to show that for some  $\varepsilon = \varepsilon_p > 0$

$$(3.36) \quad \left\| \left( \sum_{n \geq \max\{j, -j/\chi, s\}} |\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim s^{-2} 2^{-\varepsilon j} \|f\|_{\ell^p}.$$

This in turn will imply  $I_p^1 \lesssim \|f\|_{\ell^p}$  and the proof will be completed. We have assumed that  $p \geq 2$ , then for every  $g \in \ell^r(\mathbb{Z}^d)$  such that  $g \geq 0$  with  $r = (p/2)' > 1$  we have by (3.22), the Cauchy–Schwarz inequality and (3.20) that

$$\begin{aligned} (3.37) \quad \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2} \hat{f})(x)|^2 g(x) &\lesssim \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}} M_{2^n}(|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2} \hat{f})|)(x)^2 g(x) \\ &\leq \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}} M_{2^n}(|\mathcal{F}^{-1}(\Delta_{n,s}^{j,2} \hat{f})|^2)(x) g(x) = \sum_{x \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\Delta_{n,s}^{j,2} \hat{f})(x)|^2 M_{2^n}^* g(x) \\ &\lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p}^2 \left\| \sup_{N \in \mathbb{N}} M_N^* g \right\|_{\ell^r} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p}^2 \|g\|_{\ell^r}. \end{aligned}$$

Therefore, by Theorem 5 we have

$$(3.38) \quad \left\| \left( \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |\mathcal{F}^{-1}(\Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \log s \|f\|_{\ell^p}.$$

We refine the estimate in (3.38) for  $p = 2$ . Indeed, define

$$\varrho_{n,j}(\xi) = (\eta(2^{nA+jI}(\xi))^2 - \eta(2^{nA+(j+1)I}(\xi))^2) \eta(2^{s(A-\chi I)}(\xi))$$

and

$$\Psi_n(\xi) = \min \{ |2^{nA} \xi|_{\infty}, |2^{nA} \xi|_{\infty}^{-1/d}, 1 \}.$$

By Plancherel's theorem we have

$$\begin{aligned}
 (3.39) \quad & \left\| \left( \sum_{n \geq \max\{j, -j/\chi, s\}} |\mathcal{F}^{-1}(m_n \Delta_{n,s}^{j,2} \hat{f})|^2 \right)^{1/2} \right\|_{\ell^2} \\
 &= \left( \int_{\mathbb{T}^d} \sum_{n \geq \max\{j, -j/\chi, s\}} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} |m_n(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\
 &\lesssim (s+1)^{-\delta l} 2^{-|j|/(2d)} \|f\|_{\ell^2}.
 \end{aligned}$$

The last estimate is implied by (3.30). Namely, by (3.30) we may write

$$\begin{aligned}
 (3.40) \quad & \sum_{n \geq \max\{j, -j/\chi, s\}} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} |m_n(\xi)|^2 \varrho_{n,j}(\xi - a/q)^2 \\
 &\lesssim \sum_{n \geq \max\{j, -j/\chi, s\}} \sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} q^{-2\delta} (\Psi_n(\xi - a/q) + 2^{-n/2}) (2^{-|j|/d} + 2^{-n/2}) \eta(2^{s(A-\chi I)}(\xi - a/q))^2 \\
 &\lesssim (s+1)^{-2\delta l} 2^{-|j|/(2d)}.
 \end{aligned}$$

The last line follows, since we have used the lower bound for  $q \geq s^l$  if  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l$ . Moreover,

$$\sum_{n \geq 0} (\Psi_n(\xi - a/q) + 2^{-n/2}) \lesssim 1$$

and

$$\sum_{a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l} \eta(2^{s(A-\chi I)}(\xi - a/q)) \lesssim 1$$

by the disjointness of the supports of  $\eta(2^{s(A-\chi I)}(\xi - a/q))$ 's whenever  $a/q \in \mathcal{U}_{(s+1)^l} \setminus \mathcal{U}_s^l$ . Since  $l \in \mathbb{N}$  can be as large as we wish thus interpolating (3.39) with (3.38) we obtain (3.36) and the proof of (3.1) and consequently Theorem A is completed.  $\square$

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